On Utility Function for Money

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Abstract: It is well-known, that in decision analysis, it is worth to use utility function. It is a way how to make a better individual decision than in case without application of utility function. In this paper we present a way how to construct a utility function for money and how to use it for a solution of St. Petersburg paradox.

Key words: St. Petersburg paradox · Utility function for money · Expected profit

JEL Classification: C44 · D80

1 Introduction

The aim of this paper is to introduce a utility function for the money. In decision theory, it is well-known that without applying utility function, we cannot obtain relevant results. In the models of decision theory, the utility function is very often omitted, because it is easier to suppose that the utility function is linear, hence it is not necessary to do computations with it.

However, if we omit a utility function, we do not obtain such good (subjective) results as in case when we take a care about a utility function. The aim of this paper is to introduce a utility function for money, introduce the construction of this function and explain on St. Petersburg paradox a usage of utility function for money.

2 Methods

2.1 Saint Petersburg Paradox

First, let us recall a St. Petersburg paradox. St. Peters burg paradox or St. Petersburg lottery was invented by Nicolas Bernoulli in 1713. As it is known, it first appeared in a letter to Pierre Raymond de Montmort. The name of the paradox comes from the name of the city, where the resolution for this paradox was given by a cousin of Nicolas Bernoulli – by Daniel Bernoulli. Daniel Bernoulli published his arguments in the Commentaries of the Imperial Academy of Science of Saint Petersburg (today it is called Russian Academy of Science) in 1738.

St. Peters burg paradox is a problem partly from decision analysis, partly from probability theory. It is supposed a following game – game for a single player. In this game, a fair coin is tossed and it is computed the number of heads before the first tail is appeared – let us call it \( r \). Then the player wins \( 2^r \), where, as mentioned before, \( r \) is the number of tosses before the first tail appeared.

The question is what the price of this game is. It means, what is the price which is a player able to respect for this game. More precisely, what is the highest price which is the player able to pay for the possibility to play this game? (It is clear, that a player cannot lost in this game, in the worst case she/he wins only 1, but she/he can win \( 2, 4, 8, \ldots \))

Why is this problem called a paradox? The reason is that, usually, players are willing to pay maximally about \( $10 \) for a change to play this game. However, when we decide to compute the expected payoff of this game, the result is (surprisingly!) infinity. Therefore, why nobody offers for example \( $100 \) for a possibility to play this game?

Is it really an expected payoff of this game infinity? Let us compute it. As we mentioned above, there are following possible results - in case when we toss the coin and tail is first, there is no head, the payoff is \( 2^0 = 1 \). This result is achieved with probability \( 1/2 \). In case when we toss the coin and first we have head and then tail; the result is one head, the payoff is \( 2^1 = 2 \) and this result is achieved with a probability of \( 1/2 \cdot 1/2 = 1/4 \).

Hence, we can see that the possible results and their probabilities are as follows.
Table 1 Possible results in St. Petersburg Paradox

<table>
<thead>
<tr>
<th>Result</th>
<th>Number of heads</th>
<th>Probability</th>
<th>Payoff</th>
<th>Payoff * Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>0</td>
<td>1/2</td>
<td>2^0=1</td>
<td>1/2</td>
</tr>
<tr>
<td>HT</td>
<td>1</td>
<td>1/4</td>
<td>2^1=2</td>
<td>1/2</td>
</tr>
<tr>
<td>HHT</td>
<td>2</td>
<td>1/8</td>
<td>2^2=4</td>
<td>1/2</td>
</tr>
<tr>
<td>HHHHT</td>
<td>3</td>
<td>1/16</td>
<td>2^3=8</td>
<td>1/2</td>
</tr>
<tr>
<td>HHHHHT</td>
<td>4</td>
<td>1/32</td>
<td>2^4=16</td>
<td>1/2</td>
</tr>
<tr>
<td>HHHHHHT</td>
<td>5</td>
<td>1/64</td>
<td>2^5=32</td>
<td>1/2</td>
</tr>
<tr>
<td>HHHHHHHT</td>
<td>6</td>
<td>1/128</td>
<td>2^6=64</td>
<td>1/2</td>
</tr>
<tr>
<td>…</td>
<td>…</td>
<td>…</td>
<td>…</td>
<td>…</td>
</tr>
</tbody>
</table>

Now let us recall the formula for expected value. The value of payoff of this game is a discrete random variable, the well-known formula for expected value of discrete random variable is:

\[ E(V) = \sum_{i=0}^{+\infty} v_i \cdot p_i, \]

where \( i \)-s are possible results of this game, \( p_i \) is a probability of result \( i \) and \( v_i \) is a payoff in case of result \( i \). Hence, we can see from our previous table that for each \( i \), the multiplication \( v_i \cdot p_i \) is equal to \( \frac{1}{2} \), therefore the sum of \( v_i \cdot p_i \) is equal to infinity.

So, why do not players offer a higher price for this game although their expected profit is equal to infinity?

Is the calculation correct? Yes, it is. However, the convergence is very slow, for more details see Feller (1945); hence the following remarks can be raised up. First – are we able to pay all possible payoffs of this game? What is the expected payoff in case when we admit that we have only finite amount of money. Let us suppose that somebody is willing to pay up to \$2^{10} = \$1024\), respectively if he collects all his money, which he has; he can pay up to pay off after 17 heads - \$2^{17} = \$131072\). The payoff of the game after 35 heads is equal to the value of US national gold reserve which is saved at Fort Knox and after 38 heads it is equal to the value of all bank deposits in US dollars in the USA. Would the expected payoff change under these new assumptions? Let us recalculate the value of the game under these conditions – let us consider the same game, but suppose the highest possible payoffs. More precisely, we consider the same game, but in case of higher supposed payoff then it is possible, we pay only the highest possible one.

New values of the game are given in the following table.

Table 2 Values of St. Petersburg Paradox in case of finite possible result

<table>
<thead>
<tr>
<th>Possible results</th>
<th>Maximal possible payoff (US $)</th>
<th>Number of heads for this payoff</th>
<th>Value of the game (US $)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Friend</td>
<td>256</td>
<td>8</td>
<td>5</td>
</tr>
<tr>
<td>Standard</td>
<td>1024</td>
<td>10</td>
<td>6</td>
</tr>
<tr>
<td>Risk</td>
<td>131072</td>
<td>17</td>
<td>9.5</td>
</tr>
<tr>
<td>Millionaire</td>
<td>1048576</td>
<td>20</td>
<td>11</td>
</tr>
<tr>
<td>Bill Gates (2015)(^1)</td>
<td>68719476736</td>
<td>36</td>
<td>19</td>
</tr>
<tr>
<td>US golden reserve</td>
<td>8.79 \cdot 10^{12}</td>
<td>43</td>
<td>22.5</td>
</tr>
<tr>
<td>US bank deposits</td>
<td>35.18 \cdot 10^{12}</td>
<td>45</td>
<td>23.5</td>
</tr>
</tbody>
</table>

\(^1\) – Bill Gates’, US golden reserve and US bank deposits payoffs according to Wikipedia.com
Second argument, why and how to recalculate the result is to admit that the time for the game is finite. There is supposed that the game might take theoretically indefinitely (theoretically I can still have head and head and head…, one day, second day, and still head and head….), what is not doable. Hence, also in case, when we suppose that it is not a problem for us to pay every possible payoff, we should state that there is some finite number of possible tosses (= finite tome for a game).

Next argument comes from probabilities of expected payoffs – more precisely, let us suppose that we offer $X$ for the possibility to play this game. What is the probability that our payoff would be higher than $X$? For example, it is easily seen that with probability $\frac{1}{2}$, the payoff is $2$ or more, only with probability $\frac{1}{4}$, it is more than or equal to $4$; only with probability $\frac{1}{2}$, it is $8$ or more and so on. Therefore, the argument is, why to pay such value if only with such low probability I may have some profit?

The last argument, which we want to mention here, is that we do not consider a utility. If we calculate expected profit, expected payoff, we do not consider that our utility from money is not a linear function. So, let us consider the utility function. (By the way, it was a solution given by Daniel Bernoulli – application of utility function.)

2.2 Construction of utility function for money

Now, let us suppose, theoretically, that we can pay every possible payoff and we can play the game as long as is necessary. Does there exist some solution of this paradox also in this case? Yes, it is possible to solve this problem, too. We can apply a utility function for money.

In the previous analysis, we supposed that the utility function for money is linear (we do not use any transformation, so it was supposed to be linear). However, it is known, the utility function for money is a concave function. Even, by economists, it is consider that the utility function for money has a form

$$u(x) = a + b \cdot \exp(-\frac{x}{c}),$$

where $a, b, c$ are constants. Everybody has his own values of these constants. So, our aim is to find a way how to estimate values of these constants. To estimate individual constants, we first need to know at least several points from individual utility function for money.

Hence, the first task is to get some points from individual utility function for money. There are two main possible ways how to do it. First one is more complicated and gives better results; the second one is very easy but not too precise.

Estimation of parameters $a, b, c$

Let us start with the first one. First, we need to know some points from the utility function for money; it means we need to find for some amount of money the utility from this amount of money. We can suppose (it is not necessary) that the utility of $0$ is equal to zero, hence we put $u(0) = 0$. (Sometime, especially in case when we want to construct a utility function for money for some concrete lottery, where we have already decided to pay the price for the ticket to this lottery, we omit this requirement and we put utility from “–price of the ticket” is equal to zero. More precisely, we put $u(-p) = 0$, where $p$ is the price of the ticket.)

Then we need to know some other point of the utility function, usually, we want to know the highest value of the utility (usually we put it to be equal to 1, but it is not necessary, too). More precisely, in this case, we search for such amount of money that the decision maker is not able to recognize if she/he has this amount of money or if she/he has more. Mathematically, we search for $x$ such that for all $y \geq x: u(y) \approx u(x)$. And we put $u(x) = 1$ (does not matter if it is equal to 1 or to some other constant). Then we know the value of our utility function at two points. We would like to apply regression analysis to estimate values of constants $a, b, c$, so it is necessary to know more points of our utility function. To get others points, we can apply following formula

$$u(x) = p \cdot u(y) + (1 - p) \cdot u(z).$$  \hspace{1cm} (1)

What does it mean? If we know utility at points $y$ and $z$, then we can use two types of questions. First, we ask the decision maker to identify the (definitely obtaining) payoff (it would be $x$) which is indifferent (for her/him) to payoff of $y$ with probability $p$ or payoff of $z$ with probability $1 - p$. Then, we apply a formula above and receive a utility at the point $x$. 

Second possibility, how to apply previous formula is – in case, when we would like to get utility at some point \( x \) - we choose two known points of the utility function (say \( y, z \)) and put the following question. What is the value of probability \( p \) such that the decision maker is indifferent between a definitely obtaining a payoff of \( x \) or a payoff of \( y \) with probability \( p \) and a payoff of \( z \) with probability \( 1 - p \).

Hence, we can see that if we know two points of our utility function, we can generate so many other points as we need. However, if we would like to do some computations with utility function, we need to know whole function. It means to estimate the parameters \( a, b, c \). It is well known, that if we know that our utility function is in the above mentioned form, and we know several points of this function, we can apply a regression analysis to estimate parameters of the function. To finish calculations, we can use some statistical software.

**Utility function with only one parameter**

As was promised above, there exists a second – easier - way how to estimate a utility function. Now, a utility function is consider to be in the following form

\[
u(x) = R \left(1 - \exp \left(- \frac{x}{R} \right) \right)
\]

where \( R \) is an individual constant. It is easily seen, that this form of utility function is a special case of the previous one. In the previous case, it is more complicated to obtain the parameters; on the other hand the result is more precise. In this case we have only one parameter. There are two main ways how to estimate a parameter. One possibility is to apply the same method as in the previous case – to find several points of the utility function and then to use a regression analysis. However, it is quite complicated, too.

Thus, it would be better to use a special construction of utility function and search \( R \) as follows. In Hillier, Lieberman (2001), there is recommended to ask a decision maker (question (1)) to identify such \( R \) that she/he is indifferent between two following situations:

- She/he gains \( R \) dollars with probability 0.5 and lose \( \frac{R}{2} \) dollars with probability 0.5,
- Neither gain nor lose anything.

When a decision maker identifies an \( R \), we have an estimation of her/his utility function of money and we can apply it for a decision problem. The idea comes from the equation \( \frac{1}{2} u(R) + \frac{1}{2} u(-R/2) \approx 0 = u(0) \). However, if we check this equation, we can see that it fails for \( R \) too big. Hence, it seems better for us to estimate \( R \) in the following way – to ask a decision maker to identify \( M \) such that she/he cannot almost recognize if she/he has \( M \) or \( 2M \) and then put \( R = \frac{1}{10} M \).

3 Research results

3.1 Application of utility function in decision theory

When we have already estimated a utility function, or when we know it exactly, it is very easy to apply the utility function to our decision problem. It is enough to apply our utility function to all possible results of the game or experiment and to compute all expected values or decision making rules with this function. For expected values, it means that we compute an expected utility instead of expected payoff.

3.2 Calculations

In this part, let us show, how to estimate parameters of the utility function for money and then, how to apply a known (or estimated) utility function to St. Petersburg paradox.

We show how to estimate parameters of the utility function in the form (1). The estimation of parameter \( R \) was described above. To estimate parameters \( a, b, c \), we need first to know some points of our utility function. We can put \( u(0) = 0 \) and we ask a decision maker what is the highest - \( M \) - amount of money which is she/he able to recognize (in the sense that she/he is indifferent between the situation to have \( M \) dollars and the situation to have more than \( M \)).

Let us suppose that the answer is \( \$1\,000\,000 \), \( M = 1\,000\,000 \). Thus, we can put \( u(1000000) = 1 \) and now, we already know two points of a constructed utility function.
To obtain next points, we need to apply formula (1); therefore, we should ask a decision maker in following ways.

- Suppose, that there is such a game in which you may gain $100 000 with probability \( \frac{1}{2} \) or to get nothing with probability \( \frac{1}{2} \). What is the highest price which you offer for a possibility to play this game? (Sure, the answer depends on the finance situation of a decision maker, too. The solution is individual.) Let us suppose that the answer is $10 000.

In such case, we get, if we apply the equation (1): \( u(10000) = 0.5 u(1000000) + 0.5 u(0) = 0.5 \) that a utility from $10 000 is equal to \( \frac{1}{2} \).

- We continue in the same way and obtain:
  \[
  u(1000) = 0.5 u(10000) + 0.5 u(0) = 0.25,
  u(250) = 0.5 u(1000) + 0.5 u(0) = 0.125,
  u(100) = 0.5 u(250) + 0.5 u(0) = 0.25,
  u(2000) = 0.5 u(1000000) + 0.5 u(1000) = 0.75.
  \]

- Now, let us look at the points of our utility function and let us decide that we need to know the utility at the point 50000. Hence, we can rise up the following question: Suppose that you pay $50 000 for a possibility to play a game in which you may gain $1 000 000 with probability \( p \) or to get nothing with probability \( 1 - p \). What must be the value of \( p \) you would be willing to play this game? If the answer is \( p \) at least equal to 0.95, we get
  \[
  u(50000) = 0.95 u(1000000) + 0.05 u(0) = 0.95
  \]

Using the same questions and arguments, we derive
  \[
  u(5000) = 0.9 u(10000) + 0.1 u(0) = 0.45
  \text{and} u(100000) = 1 u(1000000) + 0 u(0) = 1.
  \]

Now, we know the values of the utility function at 11 points, let us summarize these result in the following table.

<table>
<thead>
<tr>
<th>Table 3 Some points of utility function</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Utility function</strong></td>
</tr>
<tr>
<td>----------------------------------------</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>100</td>
</tr>
<tr>
<td>250</td>
</tr>
<tr>
<td>1000</td>
</tr>
<tr>
<td>2000</td>
</tr>
<tr>
<td>5000</td>
</tr>
<tr>
<td>10000</td>
</tr>
<tr>
<td>20000</td>
</tr>
<tr>
<td>50000</td>
</tr>
<tr>
<td>100000</td>
</tr>
<tr>
<td>10000000</td>
</tr>
</tbody>
</table>

Hence, we can estimate the parameters \( a, b, c \). To do it, we can apply a regression analysis to estimate values of our constants \( a, b, c \). It is not a trivial problem, because it is a case of nonlinear estimation with problems of local extremes. However, we can make this problem easier, if we recall that we decided to put \( u(0) = 0 \) and \( \lim_{x \to +\infty} u(x) = 1 \). Under these conditions, we get (if \( c > 0 \))

\[
  u(0) = a + b \cdot \exp \left( \frac{-0}{c} \right) = a + b \cdot a = -b
  \]

\[
  \lim_{x \to +\infty} u(x) = a + \lim_{x \to +\infty} \exp(-x/c) = a: a = 1, \quad b = -1.
  \]

Now, it is enough to estimate a constant \( c \). To do it, we can use some statistical software or, to use an Excel add-in – Solver. The aim is to find \( c \) such that the estimated function best fit known points of the function in the sense of SSE (Sum of Square Errors). Hence, we run Solver to minimize the SSE with only one variable \( c \) (or with three variables \( a, b, c \) under the conditions that \( a = -b = 1 \)).

Fix \( a = 1, b = -1 \) and apply Solver to estimate \( c \); we get \( c = 10780.55 \).
Therefore, we obtained the utility function in the form

\[ u(x) = 1 - \exp\left(-\frac{x}{10780.55}\right). \]

Now, we can apply this utility function to St. Peterburg paradox. Let us use this function and compute an expected utility.

\[ E u(x) = \sum_{i=0}^{+\infty} u(x_i) \cdot p_i = \sum_{i=0}^{+\infty} (1 - \exp\left(-\frac{2^i}{10780.55}\right)) \cdot \frac{1}{2^{i+1}} = 0.0006728 \]

To identify the value of the game, we need to find \( x \) such that \( u(x) = 0.0006728 \). Therefore, we can derive

\[ x = -10780.55 \ln(1 - 0.0006728) = 7.26 \]

What does our result mean? We derived that even if we admit all payoffs as possible and we admit infinitely long game, the price of game should not exceed for a chosen decision maker 7.26 (for another decision maker, his utility function could differ, but the result probably would not be too different). Hence, we showed that it was worth to apply a utility function for money for a decision problem. Applying the utility function, we achieved a meaningful result.

We can also choose the second approach to the construction of the utility function, ask the decision maker the question (1), we get his \( R \) and apply the same procedure as with more complicated utility function. If we choose \( R = 10000 \) (by the construction of previous utility function, we got that utility at 100 000 is the same as utility at 1 000 000), compute the value of the game to be equal to $8.86.

4 Conclusions

In the paper we show, that it is worth to apply a utility function for money on decision-making problem. We show how to construct this utility function and we demonstrate on St. Petersburg’s paradox how to use this function on decision-making problem.

References