Approximation of a Planar B-Spline Curve by Polygonal Trail with Special Characteristics

Miloš Kaňka

Abstract: When using the output data from CAD systems, the curve is often described in a so-called “spline” format. The regulation of a cutting, respectively other machine requires, that the input data are given in a form of a sequence of linear segments with prescribed maximal difference to the real curve, expressed in the “spline” format. Specifically, we are talking about replacing the given planar B-spline curve by polygonal trail with tops on this curve. We are doing this by a way when the distance between the individual linear segments of the trail and to those linear segments corresponding parallel tangent of the curve is less than or equal to the given positive ε. The requirement of a construction like this is motivated by the fact that during the cutting process by a laser, respectively a water beam, the polygonal trail is easier to track than the whole curve.

A company MIR in Eskede, Sweden, which was the ordering party od this task, had chosen from a set of different types of planar B-spline curves the following three mostly used ones, the so-called clamped, opened and closed ones.

The positive result of solving this task become used in a regulation system AMOS, which is a component of one laser-cutting machine in a company Skanpak, Czech Republic.

Key words: Planar B-spline curves · The types of planar B-spline curves · Approximation of B-spline curves by a polygonal trail · Cox – de Boor’s formula

JEL Classification: G10 · G63 · C65

1 Introduction

In $\mathbb{R}^2$ provided with a rectangular Cartesian system of coordinates $o; x_1, x_2$, let us consider the chosen integer $p \geq 1$ configuration of $n+1$ so-called control points $P_0, P_1, ..., P_n$ (excluding the case that all of them form just one single point), and for $m = n + p + 1$ of so-called nodal vector $U = (U_0, U_1, ..., U_m)$ of dimension $m + 1$, whose components (nodes) create a non-decreasing sequence of real numbers (excluding the stationary sequence); while the $j$th node $U_{j+1}$, where $j = 1, 2, ..., m + 1$, has a so-called multiplicity $k (\geq 1, \text{integer})$, if it happens to be in this sequence $k$-times. For example, the first node of the vector $(1, 1, 1, 2, 4, 5, 6, 6)$ has a multiplicity 3, the fifth has a multiplicity 1 and the last node has a multiplicity 2. So-called (planar) B-spline curve of the $p$th degree, determined by the control points $P_r = x_i^{(r)} = (x_i^{(r)} , x_2^{(r)} )$, $r = 0, 1, ..., n$ (the index $i$ is representing the numbers 1 and 2), nodal vector $U = (U_0, U_1, ..., U_m)$ and to this vector corresponding B-spline base functions of the $p$th degree $N_{p,r}$, is defined by parametric equations:

$$x_i(u) = \sum_{r=0}^{n} P_r \cdot N_{p,r}(u) = \sum_{r=0}^{m} x_i^{(r)} \cdot N_{p,r}(u) \quad (1)$$

for $u \in <U_0, U_m>$ where $<U_0, U_m>$ is not empty (respectively $U_m$ included).

The calculation of B-spline basis functions can be realised based on Cox – de Boor’s recursive formula:

$$N_{0,j} = \begin{cases} 1 & \text{for } U_j \leq u < U_{j+1}, \\ 0 & \text{elsewhere} \end{cases}$$

$$N_{p,j} = \frac{u-U_j}{U_{p+j} - U_j} N_{p-1,j} + \frac{U_{p+j+1}-u}{U_{p+j+1} - U_{j+1}} N_{p-1,j+1} \quad (2)$$

during the calculations, the expressions that lack sense are put equal to zero.

We assign an increasing sequence of nodal points $V_0, V_1, V_2, ..., V_k$ to the nodal vector $U$.

If, for instance, the dimension of vector $U = (U_0, U_1, U_2, U_3, U_4, U_5, U_6, U_7, U_8) = (1, 1, 1, 3, 5, 8, 10, 10)$, then $V_0 = 1, V_1 = 3, V_2 = 5, V_3 = 8, V_4 = 10$. The dimension of vector $V = (V_0, V_1, V_2, V_3, V_4) = (1, 3, 5, 8, 10)$ is $sk + 1 = 4 + 1 = 5$. So, in our example, $sk = 4$. 
In applications are mostly used so-called "clamped", respectively "opened", respectively "closed" B-spline curves of degree \( p \) defined on the interval \(<U_0, U_m>\). For the clamped curve, it is requested that the initial and final nodal points of vector \( U \) have a multiplicity \( p+1 \); for opened curve, it is requested that the initial and final nodal points of vector \( U \) have a multiplicity at maximum equal to \( p \), and for closed curve, it is required that the control points for \( j = 0, 1, ..., p-1 \): \( P_j = P_{n+p+j+1} \), were true and also the coordinates of vector \( U \) form an arithmetical sequence.

In all three given cases, the constructed B-spline curve is "copying" the course of the polygonal trail with vertices \( P_0, P_1, P_2, ..., P_n \), which we call control polygonal trail. At the same time, the opened, respectively closed curve, reflects the actual state by its name, while clamped curve marks out by the fact that it has its beginning at the point \( P_0 \), and its end at the point \( P_n \), and at these points, it touches the control polygonal trail.

For \( X = U_m \), let us denote the integer from sequence \( j = 0, 1, 2, ..., sk \) by the symbol \( t \), where \( X = V_t\). Analogically for \( Y = U_m \), we denote by the symbol \( tt \) an integer from sequence \( j = 0, 1, 2, 3, ..., sk \) such that \( Y = V_{tt} \).

The number \( ss = tt - t \geq 0 \) gives the whole number (closed segments) of constructed B-spline curve. On the \( q^{th} \) segment, where \( q = 1, ..., ss \), the parameter of \( u \) is changing on the interval \(<V_{t+q}, V_{t+q}>\).

Let \( p = 3, n + 1 = 4 + 1 = 5 \) control points \( P_0, P_1, ..., P_4 \), \( m + 1 = (n + p + 1) + 1 = 8 + 1 = 9^{th} \) dimensional vector \( U = (1, 1, 1, 3, 5, 8, 8, 10, 10) \). The construction of opened B-spline curve of the third degree. To the vector \( U \) we assign a vector \( V = (V_0, V_1, V_2, V_3, V_4) = (1, 3, 5, 8, 10) \). We have \( X = U_p = U_3 = 3 = V_3 \Rightarrow t = 1 \), then we have \( Y = U_m = V_{tt} = V_8 = 8 \Rightarrow tt = 3 \). The whole number of (closed) segments of constructed curve is \( ss = tt - t = 3 - 1 = 2 \). On the 2nd segment \( (q = 2) \) the parameter is changing in interval \(<V_{t+q}, V_{t+q}>\) = \(<V_{t+1}, V_{t+2}>\) = \(<V_2, V_3>\) = \(<5, 8>\).

If for the integer \( q \) from a sequence 1, 2, ..., \( ss \) (where \( ss \geq 2 \)), it is valid that multiplicity \( k \geq 1 \) of node \( V_{t+q} \) (regards the vector \( U \)) is greater than or equal to \( p + 1 \), then (closed) segments of the curve of \( q^{th} \) and \( (q+1)^{th} \) are not continuing on each other; if \( k < p + 1 \), then they are continuing each other. In this example the first segment of the curve continues on its second segment. (see references)

2 Methods

Those methods are used in a planar B-spline curves in \( R^2 \). We also used the Cox – de Boor’s recursive formula for calculations of B-spline base functions. The algorithms for the calculation of tasks mentioned in abstract are a part of a program, that the author has created.

3 Research results

The results of this research are being used in a Swedish company MIR in a system AMOS. The AMOS is used for laser or water cutting process.

The mathematical part of solving the problem

Let us suppose that for \( i = 1, 2 \) we know parammetrical equations of \( q^{th} \) (closed) segment of constructed B-spline curve of the \( p^{th} \) degree in the form

\[
    x_i(u) = \sum_{r=1}^{p+1} a_{i}^{(r)} u^{p+1-r},
\]

where \( u \in <V_{t+q}, V_{t+q}> = I_q \) (see paragraph 1). It is the first derivative of function \( x_i(u) \):

\[
    \dot{x}_i(u) = \sum_{r=1}^{p} (p + 1 - r) a_{i}^{(r)} u^{p-r}.
\]

Let \( u_1 < u_2 \) are numbers from the interval \( I_q \), the corresponding points on studied \( q^{th} \) segment, for brevity, we denote as points \( u_1, u_2 \) (the couple of different points). The slope of secant line connecting the points \( u_1, u_2 \) is equal to division

\[
    \frac{x_2(u_2) - x_2(u_1)}{x_1(u_2) - x_1(u_1)} = k_1
\]

and the slope of tangent of the curve (exactly the \( q^{th} \) segment of the curve) in the point \( z \in (u_1, u_2) \) is equal

\[
    \frac{\dot{x}_2(z)}{\dot{x}_1(z)} = k_2.
\]

In accordance with the requirement of the task, \( k_1 = k_2 \), the following equation has to be true.
or the equation
\[
\sum_{r=1}^{p} (p+1-r) z^{p-r} \left| \begin{array}{cc} a^{(r)}_{1} & a^{(r)}_{2} \\ x_{1}(u_{2}) - x_{1}(u_{1}) & x_{2}(u_{2}) - x_{2}(u_{1}) \end{array} \right| = 0.
\] (5)

A point on the studied segment, in which the slope of the curve is parallel with the secant line connecting the points \(u_{1} < u_{2}\), it applies to an algebraic equation (5) of the degree, at most \(p - 1\). The solution of equation (5) can be the best realised with the use of Bairstow’s iteration method.

**Example 1.** For \(n = 5\), let us consider \(n + 1 = 5 + 1 = 6\) control points \(P_{0} = (1, 5), P_{1} = (5, 10), P_{2} = (10, 8), P_{3} = (6, 6), P_{4} = (15, 2), P_{5} = (8, -1)\), and for \(p = 2\) and \(m = n + p + 1 = 5 + 2 + 1 = 8\) nodal vector \(U = (U_{0}, U_{1}, ..., U_{6}) = (0, 0, 0, 3, 6, 9, 12, 12, 12)\). The vector \(V = (V_{0}, V_{1}, V_{2}, V_{3}, V_{4}) = (0, 3, 6, 9, 12)\) assigned to the vector \(U\). It will be about the construction of clamped B-spline curve of the 2nd degree. For \(X = U_{p} = U_{2} = 0 = V_{5}\) follows \(t = 0\), for \(Y = U_{m,p} = U_{6,2} = U_{6} = 12 = V_{4}\) follows \(n = 4\), so the constructed B-spline curve of the 2nd degree, designated by the chosen control points, nodal vector \(U\) and to this vector relevant B-spline base functions \(N_{ij}\) in number \(n + 1 = 6\) (the calculation of those functions according to (2))

\[
N_{i0} = \begin{cases} 
\frac{1}{9}(u^2 - 6u + 9) & \text{for } 0 \leq u < 3, \\
0 & \text{elsewhere,}
\end{cases}
\]

\[
N_{i1} = \begin{cases} 
\frac{1}{18}(3u^2 - 12u) & \text{for } 0 \leq u < 3, \\
\frac{1}{18}(u^2 - 12u + 36) & \text{for } 3 \leq u < 6, \\
0 & \text{elsewhere,}
\end{cases}
\]

\[
N_{i2} = \begin{cases} 
\frac{1}{18}u^2 & \text{for } 0 \leq u < 3, \\
\frac{1}{18}(2u^2 - 18u + 27) & \text{for } 3 \leq u < 6, \\
\frac{1}{18}(u^2 - 18u + 81) & \text{for } 6 \leq u < 9, \\
0 & \text{elsewhere,}
\end{cases}
\]

\[
N_{i3} = \begin{cases} 
\frac{1}{18}(u^2 - 6u + 9) & \text{for } 3 \leq u < 6, \\
\frac{1}{18}(2u^2 - 30u + 99) & \text{for } 6 \leq u < 9, \\
\frac{1}{18}(u^2 - 24u + 144) & \text{for } 9 \leq u < 12, \\
0 & \text{elsewhere,}
\end{cases}
\]

\[
N_{i4} = \begin{cases} 
\frac{1}{18}(u^2 - 12u + 36) & \text{for } 6 \leq u < 9, \\
\frac{1}{18}(3u^2 - 60u + 288) & \text{for } 9 \leq u < 12, \\
0 & \text{elsewhere,}
\end{cases}
\]

\[
N_{i5} = \begin{cases} 
\frac{1}{9}(u^2 - 18u + 81) & \text{for } 9 \leq u \leq 12, \\
0 & \text{elsewhere,}
\end{cases}
\]

is composed of \(ss = nt - t = 4 = 0 = 4\) (closed) segments. The parametrical equations for example the 4th (closed) segment \((q = 4)\), where the parameter \(u\) is changing in interval \(<V_{r+q-1}, V_{r+q}> = <V_{0+4-1}, V_{0+4} = <V_{3}, V_{4}> = <9, 12>\), according to (1)

\[
\begin{pmatrix} x_{1}(u) \\ x_{2}(u) \end{pmatrix} = \sum_{r=0}^{5} \begin{pmatrix} x_{1}^{(r)} \\ x_{2}^{(r)} \end{pmatrix} N_{2,r}(u) = \begin{pmatrix} 1 \end{pmatrix} \cdot 0 + \begin{pmatrix} 5 \\ 10 \end{pmatrix} \cdot 0 + \begin{pmatrix} 10 \\ 8 \end{pmatrix} \cdot 0 + \begin{pmatrix} 6 \\ 18 \end{pmatrix} \cdot (u^2 - 24u + 144) + \begin{pmatrix} 15 \\ 18 \end{pmatrix} \cdot (-3u^2 + 60u - 288) + ...
\]

(6)
For \( u = 12 \), according to (6) we get the ending point of the curve \( P_5 = (8, -1) \). Are parametrical equations of the hodograph of the curve (of it’s 4th segment)

\[
\begin{pmatrix}
\hat{x}_1(u) \\
\hat{x}_2(u)
\end{pmatrix} = \begin{pmatrix}
-\frac{1}{9} (23u - 23) \\
-\frac{1}{9} (2u + 6)
\end{pmatrix},
\]

and so we have

\[
\begin{pmatrix}
\hat{x}_1(12) \\
\hat{x}_2(12)
\end{pmatrix} = \begin{pmatrix}
-14/3 \\
-2
\end{pmatrix}, \quad \frac{\hat{x}_1(12)}{\hat{x}_2(12)} = \frac{-2}{-14/3} = \frac{3}{7}.
\]

And a line segment \( P_4P_5 \) of control polygonal trail has a tangent

\[
\frac{2 - (-1)}{15 - 8} = \frac{3}{7}.
\]

The curve is touching the control polygonal trail in the point \( P_5 \).

For \( q = 3 \) has a node \( V_t+q = V_0+3 = U_5 \) of multiplicity \( k = 1 \) and it is true that \( k = 1 < p + 1 = 2 + 1 = 3 \). So the 3\(^{rd} \) and the 4\(^{th} \) (closed) segments of the constructed curve are linked to each other. For the considered example we will make an equation (5). According to (6) is

\[
a_1^{(1)} = -23/18, \quad a_1^{(2)} = 468/18 = 26, \tag{7}
\]

\[
a_2^{(1)} = -1/9, \quad a_2^{(2)} = 6/9 = 2/3,
\]

and then

\[
\begin{align*}
x_1(u_2) &= x_1(12) = 8, \quad x_1(u_1) = x_1(9) = 21/2, \\
x_2(u_2) &= x_2(12) = -1, \quad x_2(u_1) = x_2(9) = 4,
\end{align*}
\]

so the equation (5) with the use of (7) and (8) will have the form

\[
0 = 2z \begin{vmatrix} -23/18 & -1/9 \\ -5/2 & -5 \end{vmatrix} + \begin{vmatrix} 26 & 2/3 \\ -5/2 & -5 \end{vmatrix} = 2z \cdot \frac{55}{9} - \frac{385}{3} = z = \frac{385/3}{110/9} = \frac{385 \cdot 9}{110 \cdot 3} = \frac{385}{110} = \frac{21}{2}, \tag{9}
\]

A point \( z \in (9, 12) \), in which the tangent of the curve is parallel with secant, connecting the points \( u_1 = 9 < 12 = u_2 \), is equal to a number \( z = 21/2 \).

**Example 2.** The problem is the same as in the Example 1. The determinant in equation (5)

\[
D_r = \begin{vmatrix} a_1^{(r)}(u_2) & a_1^{(r)}(u_1) \\ x_1(u_2) - x_1(u_1) & x_2(u_2) - x_2(u_1) \end{vmatrix}
\]

it can be written in a form \( \epsilon_{ij} a_i^{(r)}(x(u_2) - x(u_1)) \), where \( i, j \) are addition indexes changing independently on each other from 1 to 2 (Einstein summation convention), where \( \epsilon_{ij} \) is Levi-Civita’s tenzor of the 2\(^{nd} \) degree with coordinates

\[
\epsilon_{ij} = \begin{cases} 
0, & \text{for } i = j, \\
1, & \text{for } i < j, \\
-1, & \text{for } i > j,
\end{cases}
\]

as

\[
x_i(u_2) - x_i(u_1) = a_j^{(1)}(u_2^2 - u_1^2) + a_j^{(2)}(u_2 - u_1),
\]
we have
\[ D_r = (u_x^2 - u_x^1) \cdot \epsilon_{ij} \cdot a_i^{(r)} \cdot a_j^{(r)} + (u_z - u_t) \cdot \epsilon_{ij} \cdot a_i^{(r)} \cdot a_j^{(r)}. \]

For \( r = 1 \) is
\[ D_1 = (u_x^2 - u_x^1) \cdot \epsilon_{ij} \cdot a_i^{(1)} \cdot a_j^{(1)} + (u_z - u_t) \cdot \epsilon_{ij} \cdot a_i^{(1)} \cdot a_j^{(1)}, \]
and because
\[ \epsilon_{ij} \cdot a_i^{(1)} \cdot a_j^{(1)} = \epsilon_{ij'} \cdot a_i^{(1)} \cdot a_j^{(1)} = - \epsilon_{ij} \cdot a_i^{(1)} \cdot a_j^{(1)}, \]
is \( \epsilon_{ij} \cdot a_i^{(1)} \cdot a_j^{(1)} = 0 \) and because
\[ D_1 = (u_z - u_t) \cdot \epsilon_{ij} \cdot a_i^{(1)} \cdot a_j^{(1)}. \]

Further we have
\[ D_2 = (u_x^2 - u_x^1) \cdot \epsilon_{ij} \cdot a_i^{(2)} \cdot a_j^{(2)} + (u_z - u_t) \cdot \epsilon_{ij} \cdot a_i^{(2)} \cdot a_j^{(2)}, \]
and because again \( \epsilon_{ij} \cdot a_i^{(2)} \cdot a_j^{(2)} = 0 \), is
\[ D_2 = (u_x^2 - u_x^1) \cdot \epsilon_{ij} \cdot a_i^{(1)} \cdot a_j^{(1)}. \]

The equation (5) is then
\[ 0 = 2z \cdot D_1 + D_2 = 2z \cdot (u_z - u_t) \cdot \epsilon_{ij} \cdot a_i^{(1)} \cdot a_j^{(1)} + (u_z - u_t) \cdot (u_z - u_t) \cdot \epsilon_{ij} \cdot a_i^{(2)} \cdot a_j^{(2)}, \]
that is
\[ 0 = 2z \cdot \epsilon_{ij} \cdot a_i^{(1)} \cdot a_j^{(2)} + (u_z - u_t) \cdot \epsilon_{ij} \cdot a_i^{(2)} \cdot a_j^{(1)} \]  \hspace{1cm} (10)

and because
\[ \epsilon_{ij} \cdot a_i^{(2)} \cdot a_j^{(1)} = \epsilon_{ij'} \cdot a_i^{(2)} \cdot a_j^{(1)} = - \epsilon_{ij} \cdot a_i^{(1)} \cdot a_j^{(2)}, \]
(9) could be written in a form
\[ 0 = \epsilon_{ij} \cdot a_i^{(1)} \cdot a_j^{(2)}. \]  \hspace{1cm} (11)

Supposing that the determinant
\[ \epsilon_{ij} \cdot a_i^{(1)} \cdot a_j^{(2)} = \begin{vmatrix} a_i^{(1)} & a_j^{(2)} \\ a_i^{(2)} & a_j^{(1)} \end{vmatrix} \neq 0, \]  \hspace{1cm} (12)
follows (10)
\[ Z = \frac{u_x^1 + u_x^2}{2} \]  \hspace{1cm} (13)

For \( u_1 = 9 < 12 = u_2 \) follows from (12), that \( z = (9 + 12)/2 = 21/2 \), which corresponds to the result (8) from Example 1.

If we choose in interval <9, 12> \( I_1 \) for example points \( u_1 = 10 < 12 = u_2 \), then based on (12)
\[ Z = \frac{10+12}{2} = \frac{22}{2} = 11. \]

In this point is the tangent of the curve parallel to the secant connecting those two points. The determinant (11) with elements (7) is not equal to zero, how we can easily find out.

**Example 3.** For \( n = 9 \) let us study \( n + 1 = 9 + 1 = 10 \) control points \( P_0 = (1, 11), P_1 = (12, 11), P_2 = (12, 2), P_3 = (8, 2), P_4 = (8, 8), P_5 = (4, 8), P_6 = (4, 3), P_7 = (1, 3), P_8 = (1, 11), P_9 = (12, 11), \) and for \( p = 2 \) and \( m = n + p + 1 = 9 + 2 + 1 = 12 \) nodal vector \( U = (U_0, U_1, U_2, ..., U_{12}) = (0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12), \) the components of which create an arithmetical sequence. A vector \( V \) assigned to the vector \( U \) is equal to the vector \( U \).

For \( j = 0, 1 = 2 - 1 = p - 1 \) is true. \( P_j = P_{n+p+j}: P_0 = P_{9+2+0+1} = P_9, P_1 = P_{9+2+1+1} = P_9. \) It will be a construction of a closed B-spline curve of the 2\(^{nd} \) degree. For \( X = U_p = U_2 = 2 = V_2 \) follows \( t = 2 \), for \( Y = U_{n+p} = U_{10} = 10 = V_{10} \) follows \( t = 10 \), so the constructed B-spline curve of the 2\(^{nd} \) degree, determined by the given control points, nodal vector \( U \) and to this vector relevant B-spline base functions \( N_{2j} \) for \( j = 0, 1, ..., 8 \).
The tangent of the secant, connecting the side points of interval \( I_n \), which are points \( u_1 = 9 < 10 = u_2 \), is equal
\[
\frac{x_2(10) - x_2(9)}{x_1(10) - x_1(9)} = \frac{11 - 7}{11 - 1} = \frac{8}{11} = k_1,
\]
and the tangent of secant of the curve in a point \( z = (u_1 + u_2)/2 = (9 + 10)/2 = 19/2 \) (see Example 2) is equal to
\[
\frac{x_2(19/2)}{x_1(19/2)} = \frac{(-8u + 80)_{u = 19/2}}{(11u - 99)_{u = 19/2}} = \frac{-152 + 160}{265 - 198} = \frac{4}{11} = k_2,
\]
so \( k_1 = k_2 \). In a point \( z = 19/2 \in (9, 10) \) is a tangent of the curve parallel with secant, connecting the points \( u_1 = 9 < 10 = u_2 \).

**Example 4.** For \( n = 4 \) let us consider \( n + 1 = 4 + 1 = 5 \) control points: \( P_0 = (1, 5), P_1 = (5, 10), P_2 = (10, 8), P_3 = (6, 6), P_4 = (15, 2) \), and for \( p = 3 \) and \( m = n + p + 1 = 4 + 3 + 1 = 8 \) nodal vector
\[
U = (U_{06}, U_1, U_2, ..., U_7, U_b) = (0, 5, 5, 5, 10, 10, 15, 20).
\]
A vector \( V = (V_0, V_1, V_2, V_3, V_4) = (0, 5, 10, 15, 20) \) is associated to the vector \( U \). This will be about a construction of opened B-spline curve of the 3rd degree. For \( X = U_\rho = U_5 = V_1 \) follows \( t = 1 \), for \( Y = U_{m,p} = U_3 = 10 = V_2 \) follows \( t = 2 \), so the constructed B-spline curve of the 3rd degree, designated by given control points, nodal vector \( U \) and to this vector associated B-spline base functions \( N_{3,j} \) in number \( n + 1 = 5 \) (the calculation of those functions according to (2))
\[
N_{3,j} = \begin{cases} 
\frac{1}{125} u^3 & \text{for } 0 \leq u < 5, \\
0 & \text{elsewhere}, 
\end{cases}
\]
is composed of \( ss = tt = 2 - 1 = 1 \) (closed) segment. The parametrical equations of the first (closed) segment \((q = 1)\), on which, the parameter \( u \) is changing in interval \( <V_{i+q-1}, V_{i+q}> = <V_{i+1}, V_q> = <V_1, V_2> = <5, 10> = I_i \), are equal (the calculation based on (1))

\[
x_1(u) = \frac{7u^3 - 159u^2 + 1215u - 2725}{50},
\]

\[
x_2(u) = \frac{-6u + 80}{5}.  \tag{16}
\]

Based on (16) we have

\[
a_1^{(1)} = \frac{7}{50}, \quad a_1^{(2)} = -\frac{159}{50}, \quad a_2^{(3)} = -\frac{6}{5},
\]

\[
a_2^{(1)} = 0, \quad a_2^{(2)} = 0, \quad a_2^{(3)} = -\frac{6}{5}, \tag{17}
\]

then for \( u_1 = 6 < 9 = u_2 \) (a couple of different points on studied segment)

\[
x_1(u_2) = x_1(9) = 217/25, \quad x_1(u_1) = x_1(6) = 353/50,
\]

\[
x_2(u_2) = x_2(9) = 26/5, \quad x_2(u_1) = x_2(6) = 44/5, \tag{18}
\]

so the equation (5) will be with the use of (17), (18) of a shape

\[
0 = 3z^2 \left[ \begin{array}{c}
\frac{7}{50} & 0 \\
\frac{81}{50} & -\frac{18}{5}
\end{array} \right] + 2z \left[ \begin{array}{c}
\frac{159}{50} \\
\frac{81}{50}
\end{array} \right] + \left[ \begin{array}{c}
\frac{243}{10} \\
\frac{81}{50}
\end{array} \right] = \frac{6}{5},
\]

which can be written in a form

\[
7z^2 - 106z + 396 = 0.
\]

The solutions of this quadratic equation are

\[
z_{1,2} = \frac{106 \pm \sqrt{106^2 - 4 \cdot 7 \cdot 396}}{14} = \frac{53 \pm \sqrt{37}}{7} = \{8.4404, 6.7025\},
\]

and both are elements of interval \((u_1, u_2) = (6, 9)\).

The slope of the tangent, connecting the points \( u_1 = 6 < 9 = u_2 \), is according to (18)

\[
x_2(9) - x_2(6) = \frac{26/5 - 44/5}{434/50 - 353/50} = \frac{-20}{9} = k_1,
\]

The slope of a tangent to the curve in points \( z_{1,2} \) is equal to

\[
x_2(z_{1,2}) = \left( \frac{-6/5}{21u^2 - 318u + 1215} \right)_{u=z_{1,2}} = \frac{-20}{9} = k_2.
\]

So \( k_1 = k_2 \), as required.
Example 5. For \( n = 4 \) let us consider \( n + 1 = 4 + 1 = 5 \) control points \( P_0 = (1,5), P_1 = (5,10), P_2 = (10,8), P_3 = (6,6), P_4 = (15,2) \), and for \( p = 4 \) and \( m = n + p + 1 = 4 + 4 + 1 = 9 \) nodal vector \( U = (U_0, U_1, U_2, ..., U_8, U_9) = (0, 0, 0, 1, 2, 3, 3, 3, 3) \). A vector \( V = (V_0, V_1, V_2, V_3) = (0, 1, 2, 3) \) is associated to the vector \( U \). For \( X = U_{t+1} = 1 = V_1 \) follows \( t = 1 \), for \( Y = U_{m+p} = U_8 = 2 = V_2 \) follows \( n = 2 \), so the constructed, opened B-spline curve of the \( 4^{th} \) degree, designated by given control points, nodal vector \( U \) and to this vector relevant B-spline base functions \( N_{4,t} \) in quantity \( n+1 = 5 \) (the calculation of those functions based on (2)).

\[
N_{4,0} = \begin{cases}
-\frac{1}{8} (15 u^4 - 56 u^3 + 72 u^2 - 32 u) & \text{for } 0 \leq u < 1, \\
\frac{1}{8} (u^4 - 8 u^3 + 24 u^2 - 32 u + 16) & \text{for } 1 \leq u < 2, \\
0 & \text{elsewhere},
\end{cases}
\]

\[
N_{4,1} = \begin{cases}
\frac{1}{72} (85 u^4 - 264 u^3 + 216 u^2) & \text{for } 0 \leq u < 1, \\
\frac{1}{72} (23 u^4 - 168 u^3 + 432 u^2 - 432 u + 108) & \text{for } 1 \leq u < 2, \\
\frac{1}{10} (u^4 - 12 u^3 + 54 u^2 - 108 u + 81) & \text{for } 2 \leq u < 3, \\
0 & \text{elsewhere},
\end{cases}
\]

\[
N_{4,2} = \begin{cases}
-\frac{1}{36} (13 u^4 - 24 u^3) & \text{for } 0 \leq u < 1, \\
\frac{1}{36} (14 u^4 - 84 u^3 + 162 u^2 - 108 u + 27) & \text{for } 1 \leq u < 2, \\
-\frac{1}{36} (13 u^4 - 132 u^3 + 486 u^2 - 756 u + 405) & \text{for } 2 \leq u < 3, \\
0 & \text{elsewhere},
\end{cases}
\]

\[
N_{4,3} = \begin{cases}
\frac{1}{72} (23 u^4 - 108 u^3 + 162 u^2 - 108 u + 27) & \text{for } 0 \leq u < 1, \\
\frac{1}{72} (85 u^4 - 756 u^3 + 2430 u^2 - 3348 u + 1701) & \text{for } 1 \leq u < 2, \\
0 & \text{elsewhere},
\end{cases}
\]

\[
N_{4,4} = \begin{cases}
\frac{1}{8} (u^4 - 4 u^3 + 6 u^2 - 4 u + 1) & \text{for } 1 \leq u < 2, \\
-\frac{1}{8} (8 u^4 - 61 u^3 + 168 u^2 - 200 u + 87) & \text{for } 2 \leq u \leq 3, \\
0 & \text{elsewhere},
\end{cases}
\]

consists of \( x_5 = u - t = 2 - 1 = 1 \) (closed) segment. The parametrical equations of this first (closed) segment \( (q = 1) \), on which the parameter \( u \) is changing in interval \( <V_{i+q}, V_{i+q}> = <V_{i+1}, V_{i+1}> = <V_i, V_2> = <1, 2> = I_i \), are equal (calculation based on (1))

\[
\begin{pmatrix} x_1(u) \\ x_2(u) \end{pmatrix} = \sum_{r=0}^{4} P_r \cdot N_{4,r}(u) = \\
= \left( \frac{1}{5} \right) \frac{u^4 - 8 u^3 + 24 u^2 - 32 u + 16}{8} + \left( \frac{5}{10} \right) \frac{-23 u^4 + 168 u^3 - 432 u^2 + 432 u - 108}{72} + \\
+ \left( \frac{6}{10} \right) \frac{14 u^4 - 84 u^3 + 162 u^2 - 108 u + 27}{36} + \left( \frac{6}{2} \right) \frac{-23 u^4 + 108 u^3 - 162 u^2 + 108 u - 27}{72} + \\
+ \left( \frac{15}{2} \right) \frac{u^4 - 4 u^3 + 6 u^2 - 4 u + 1}{8} = \\
= \left( \frac{57 u^4 - 268 u^3 + 378 u^2 - 60 u + 39}{24} \right) - \left( \frac{81 u^4 - 552 u^3 + 1512 u^2 - 1728 u + 72}{24} \right).
\]

For the points \( u_1 = 1 < 3/2 = u_2 \), laying in interval \( I_1 \), is based on (19)
\( x_i(u_i) = x_i(1) = 146/24 = 73/12 \), \( x_i(u_2) = x_i(3/2) = 979/128 \).
Approximation of a Planar B-Spline Curve by Polygonal Trail with Special Characteristics

\[ x_1(u_1) = x_2(1) = 615/24 = 205/8, \quad x_2(u_2) = x_3(2/3) = 3045/128, \]

which is

\[
\begin{align*}
    x_1(3/2) - x_1(1) &= \frac{979}{128} - \frac{73}{12} = \frac{601}{384}, \\
    x_2(3/2) - x_2(1) &= \frac{979}{128} - \frac{73}{12} = \frac{3045}{128} - \frac{205}{8} = -\frac{235}{128},
\end{align*}
\]

so the slope of a secant, connecting the points \( u_1 = 1 < 3/2 = u_2 \), is equal to

\[
    \frac{x_2(3/2) - x_2(1)}{x_1(3/2) - x_1(1)} = -\frac{235/128}{705/601} = -1.173 = k_1.
\]

For the considered case, we will construct an equation (5). Based on (19) is

\[
    a_1^{(1)} = \frac{57}{24} = \frac{19}{8}, \quad a_1^{(2)} = -\frac{269}{24} = -\frac{67}{6}, \quad a_1^{(3)} = \frac{378}{24} = \frac{63}{4}, \quad a_1^{(4)} = -\frac{60}{24} = -\frac{5}{2},
\]

\[
    a_2^{(1)} = -\frac{81}{24} = -\frac{27}{8}, \quad a_2^{(2)} = \frac{552}{24} = 23, \quad a_2^{(3)} = -\frac{1512}{24} = -63, \quad a_2^{(4)} = \frac{1728}{24} = 72,
\]

so based on (20), (21) (considering the appropriate modifications of the determinants)

\[
\]

and after dividing by the number 3, we have

\[
    0 = 944z^3 - 11901z^2 + 35679z - 27673.
\]

Using Bairstow’s iterative method, a cubic equation (22) has three real roots (with an accuracy of 4 decimal places)

\[
    2.7591, 1.2334, 8.6146.
\]

From which, only the second one, that is 1.2334, is laying in interval \((u_1, u_2) = (1, 3/2)\). By derivation of the equation (19), we get

\[
    \begin{align*}
        \dot{x}_1(u) &= \frac{228u^3 - 804u^2 + 756u - 60}{24} = \frac{57u^3 - 201u + 189u - 15}{6}, \\
        \dot{x}_2(u) &= \frac{-324u^3 + 1656u^2 - 3024u + 1728}{24} = \frac{-81u^3 + 414u^2 - 756u + 432}{6},
    \end{align*}
\]

so the slope of tangent of the curve in point \( z = 1.2334 \) is equal to

\[
    \begin{align*}
        \dot{x}_1(1.2334) &= \frac{1}{6}(-81u^3 + 414u^2 - 756u + 432)_{u=1.2334} = -1.173 = k_2, \\
        \dot{x}_1(1.2334) &= \frac{1}{6}(57u^3 - 201u^2 + 189u - 15)_{u=1.2334} = 1.173 = k_2.
    \end{align*}
\]

So \( k_1 = k_2 \), which means, that tangent of the curve in point \( z = 1.2334 \) is parallel to a secant connecting the points \( u_i = 1 < 3/2 = u_2 \).

The construction of required polygonal trail

Primarily, let us choose a positive number \( \varepsilon \) and an integer \( H \geq 5 \). The interval \( I_1 = <u_1 = V_1, V_{c1} = u_2> \), where the parameter \( u \) is changing on the first (closed) segment of constructed B-spline curve of the \( p \) degree \( (q = 1) \), we will divide (for example equally) to \( H \) partial intervals using the dividing points

\[
    u_i = d_0 < d_1 < d_2 < \ldots < d_{H-1} = u_2.
\]
We will try to look for touching points of tangent curves \( z_1^{(j)}, z_2^{(j)}, \ldots, z_k^{(j)} \), for integer \( j = 1, 2, \ldots, H \), which are parallel with secant connecting the point \( u_1, d_j \) (because the degree of algebraic equation (5) is less than or equal to \( p-1 \)).

For the total number of points \( k \), it is true, that inequality \( k \leq p - 1 \) is valid for all numbers \( k \). We will continue in this process of searching until the maximum of numbers \( \epsilon_1^{(j)}, \epsilon_2^{(j)}, \ldots, \epsilon_k^{(j)} \), giving the distance between the relevant tangent of the curve and the secant of the curve, will be less than or equal to the chosen number \( \epsilon \); for the last number \( j = f \) we will choose a point \( d \) instead of the end point of a line segment, which has the beginning in the point \( u_1 \). In the next step, we will move the point \( d_j \) to the place of the point \( u_1 \) (as it is common in creating computer programs), and in interval \( (d_1, u_\cap) \) we will continue analogically as in the first case. We will stop when the inequality \( d_1 \geq u_2 \) is true. The result of this type of processing of the first segment of the curve will be a certain sequence of parameters

\[
u_1 = d_0 < d_1 < d_2 < \ldots < d_{j_w} < u_2,
\]

to which the corresponding points on the first segment of the curve will form tops of polygonal trail, meeting formulated task (see abstract).

A situation, where the original choice of the numbers \( \epsilon \) and \( H \) is unsuitable, could happen (from the point of view of given assignment), and it is therefore necessary to change it.

After processing the first segment of B-spline curve of the \( p \)th degree, we will continue in an analogous way when processing the second (closed) segment of the curve (if it exists), realised in a process of choosing two numbers \( \epsilon > 0 \) and integer \( H \geq 5 \), which does not have to be the same as in the first segment, etc. The polygonal trail, constructed of individual polygonal trails of particular segments of the curve, then meets the specified task for the number \( \epsilon \), which is the maximal out of numbers \( \epsilon \) for each segments.

It is obvious that the described algorithm of the solution of given assignment requires converting the assignment into computer program. The author of this article created this program.

**Example 6.** For \( n = 5 \), let us consider \( n + 1 = 5 + 1 = 6 \) control points as in Example 1, and for \( p = 3 \) and \( m = n + p + 1 = 5 + 3 + 1 = 9 \), nodal vector \( U = (U_0, U_1, U_2, \ldots, U_6, U_7) = (0, 0, 1, 4, 6, 7, 8, 8, 8) \). A vector \( V = (V_0, V_1, V_2, V_3, V_4, V_5) \) is associated to the vector \( U \). This will be about a construction of opened B-spline curve of the 3rd degree. For \( X = U_p = U_3 = 1 \) follows \( t = 1 \), for \( Y = U_{m\cdot p} = U_6 = 7 = V_4 \) follows \( t = 4 \), so the constructed B-spline curve of the 3rd degree, designated by given control points, nodal vector \( U \) and to this vector relevant B-spline base functions \( N_{3,3} \) is composed of \( s = n - t - 4 = 1 \) (closed) segments. Segments meet each other. The parametrical equations of the first (closed) segment \( (q = 1) \) are

\[
x_1(u) = \frac{-31u^2 + 48uv^2 + 1032u + 16}{360},
\]

\[
x_2(u) = \frac{95u^2 - 1140uv^2 + 3840uv - 2320}{720},
\]

where \( u \in \langle V_{i\cdot q}, V_{i+q} \rangle = \langle V_1, V_2 \rangle = \langle 1, 4 \rangle = I_1 \). For example, a computer generated sequence (23) for \( \epsilon = 0.05 \) and \( H = 8 \) is following:

\[
\]

The points on the first segment of a curve, corresponding to the tops of the polygonal trail we are looking for are based on (24) and (25). To be sure, for instance we can choose a couple of points \( u_1 = 3.058 < 3.529 = u_2 \), from this sequence of parameters, and construct an equation of type (5), we will obtain a quadratic equation

\[
1.676\epsilon^2 + 2(1.103)\epsilon - 25.461 = 0.
\]

Its root, laying between 3.058 and 3.529, is equal to 3.296. The distance of this point, on the first segment of curve with this parameter, that is a touching point of tangent of the curve (which is parallel with a secant containing these two points), is 0.038, that is less than or equal to \( \epsilon = 0.05 \).

The parametrical equations of the second (closed) segment of the constructed curve \( (q = 2) \) are

\[
x_1(u) = \frac{229u^3 - 3072u^2 + 13512u - 16624}{360},
\]

\[
x_2(u) = \frac{-u^3 + 12u^2 - 57u + 169}{9},
\]
where $u \in <V_{t+q-j}, V_{t+q}> = <V_2, V_3> = <4, 6> = I_2$. For example, a computer generated sequence (23) for $\varepsilon = 0.05$ and $H = 8$ is

$4 < 4.75 < 5.063 < 5.414 < 5.78 < 6$.

The points on the second segment of a curve, based on (26) and (27), are corresponding to the tops of the polygonal trail we are looking for.

To be sure, for instance we can choose a couple of points $u_1 = 4 < 4.75 = u_2$, from this sequence of parameters, and we can also construct an equation of the type (5), we will obtain a quadratic equation

$$0.264z^2 - 2 \cdot (1.171)z + 5.179 = 0.$$  

Its two roots: $z_1 = 4.21$, $z_2 = 4.659$, are laying between 4 and 4.75. The distance of this point, on the second segment of the curve, with parameter $z_1$, that is a touching point of tangent of the curve (which is parallel with a secant containing these two points 4 < 4.75), is $\varepsilon_1 = 0.024$; for an analogous case, a point with parameter $z_2$ is $\varepsilon_2 = 0.004$. The max{$\varepsilon_1, \varepsilon_2$} = 0.024 ≤ 0.05.

The parametrical equations of the third (closed) segment of constructed curve ($q = 3$) are

\begin{align*}
 x_1(u) &= \frac{-277u^3 + 5196u^2 - 32160u + 66400}{72}, \quad (28) \\
 x_2(u) &= \frac{-u^3 - 6u^2 + 96u + 28}{36}, \quad (29)
\end{align*}

where $u \in <V_{t+q-j}, V_{t+q}> = <V_3, V_4> = <6, 7> = I_3$. For example, a computer generated sequence (23) for $\varepsilon = 0.05$ and $H = 8$ is

$6 < 6.5 < 6.638 < 6.844 < 7$.

The points on the third segment of a curve, based on (28) and (29), are corresponding to the tops of the polygonal trail we are looking for.

To be sure, for instance we can choose a couple of points $u_1 = 6.844 < 7 = u_2$, from this sequence of parameters, and construct an equation of type (5), we will obtain a quadratic equation

$$2.011z^2 - 2 \cdot (12.592)z + 77.986 = 0.$$  

Its root, laying between 6.844 and 7, is equal to 6.915. The distance of this point, on the third segment of curve with this parameter, that is a touching point of tangent of the curve (which is parallel with a secant containing these two points), is 0.046, that is less than or equal to $\varepsilon = 0.05$.

A polygonal trail, constructed out of partial polygonal trails for each segments of the curve, is fulfilling the assignment for $\varepsilon = 0.05$.

### 4 Conclusions

An approximation of the B-spline curves of the $p^{th}$ degree ($p \geq 1$, integer) by a polygonal trail is described in this article. The approximation of the curve is fulfilling the assignments form abstract. Using a computer program, that the author has created, we solved a problem in Example 6.

### References


