Methodology of Theoretical Physics in Economics: Vector Theory of Retail Gravitation Law

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Abstract: We assume that non-satiation axioms are general economic axioms which are associated with the genetic essence of life existing in any part of the Universe. Maximizing utility under given initial and boundary conditions is the foremost interest of every individual. Genetically coded into this maximizing of utility is the survival instinct of both the given individual and the species as a whole. The noosphere continually manifests through the geosphere and biosphere in the form of human interventions in these, and is visibly represented by the physical and economic development of the Earth. One of the many phenomena which may be used to characterize the United States in the late nineteenth and early twentieth centuries is the rapid change which occurred in retail trade relations in various parts of the country. The common feature of these changes was the flow of retail business from small towns to large cities. However, no general analytical laws were known to describe the rise and distribution of this flow of retail business in space and time. From 1927-1930 W. J. Reilly conducted a nationwide study of retail dynamics. One of the findings of this study was the scalar law of retail gravitation. This law of Reilly considered the unidirectional flow of retail trade from small towns to cities. The reverse flow of retail trade from large cities to small towns was not considered because at the time it was far less significant compared to the flow of retail trade from small towns to large cities. Reilly’s scalar one-dimensional model of retail gravitation is generalized in the three-dimensional vector model of retail gravitation for the geoid. The scalar potential of retail gravitation is introduced along with the vector of an intensity of retail gravitation.

Key words: Consilience · Law of Inertia · Law of Force · Law of Interaction · Law of Gravitation · Law of Retail Gravitation · Space Economics

JEL Classification: A12 · C65

1 Introduction

The relationship which describes a possible movement of a body in space is called an equation of motion. We understand a body to be a body or particle or possibly a set of particles. Space is understood to be the forces and fields of force acting upon the body and the constraints which limit its motion. By solving the equation of motion, we obtain the position of the body at any given moment. In classical mechanics, the solution describes the trajectory of the body. In quantum mechanics it is the result of a time-varying wave function.

In their most general form, equations of motion are typically differential equations of the second order, where we take the derivation of time. The solution is a vector function describing the position of the body in relation to time \( \vec{r} = \vec{r}(t) \), which is an expression of the trajectory of the end point of the vector radius. If \( m = \text{const.} \), the basic dynamics equation takes the form

\[
\vec{F} = m \vec{a} = m \frac{d\vec{v}}{dt} = m \frac{d^2 \vec{r}}{dt^2},
\]

resultant force \( \vec{F} \) and acceleration \( \vec{a} \) of the rigid body are vectors in the same direction. We call this a motion equation. For force, we substitute the functions of time, position or velocity, i.e. in general \( \vec{F} = \vec{F}(\vec{r}, \vec{v}, t) \).

With respect to consilience in economics and physics as well as social sciences and physics, physics is presented with the question of whether it is possible to derive Reilly’s law of retail gravitation from Newton’s law of gravitation. In this paper we will attempt to provide a brief answer to this question.

2 Theoretical background

The universal law of gravitation formulated by Isaac Newton (Newton, 1960) can be expressed in the modern language of physics as: „every body of the universe attracts every other body towards its centre of gravity, with a force, which
is proportional to the product of the gravitational masses of the bodies and inversely proportional to the square of the distance between them. For mass points at a distance \( r \) with gravitational masses \( M_1 \) and \( M_2 \), the law of universal gravitation is expressed in the form

\[
F_g = \kappa \frac{M_1 M_2}{r^2},
\]

where \( F_g \) is the value of the force of attraction between the two mass points. The constant of proportionality \( \kappa \) is the universal gravitational constant. The value of the universal gravitational constant is as follows:

\[
\kappa = 6.67408 \times 10^{-11} \text{ m}^3 \text{kg}^{-1} \text{s}^{-2}.
\]

The word gravitation is derived from the Latin gravitatem, i.e. mass, gravity, and the law of retail gravitation states that the amount of retail trade which a large city attracts is dependent on the number of inhabitants of that large city and the distance of the small town from which the trade is drawn as well as on the number of inhabitants of the small city. Let us attempt to explain the significance of this claim using a model. Let us assume that on a Euclidean plane there are two cities with different populations. The number of inhabitants \( P_B \) of city \( B \) is much smaller than the number of inhabitants \( P_A \) of city \( A \). Let \( r_{AB} \) be the shortest distance between the borders of the two self-governing cities. The large city \( A \) attracts retail trade from town \( B \) to the territory of the city \( A \). The amount of retail trade attracted from town \( B \) to city \( A \) is directly proportional to the product \( P_B \cdot P_A \) of the number of inhabitants of both cities and inversely proportional to the square of the distance between the cities \( r_{AB} \). This correlates directly with Newton’s Law of Gravitation. That is why Reilly’s law is called "law of retail gravitation".

Everyone knows two qualitative laws which describe retail flow from smaller cities and towns to larger cities. The first law states that, under similar conditions, the larger a city is the greater external retail trade it attracts (Reilly, 1931, p. 7). The second law states that a large city attracts more retail trade from closer towns than from more distant towns (Reilly, 1931, p. 7).

Let us assume that long-term economically active adult inhabitants of the attracting large city have a dominant influence on attracting external retail trade to the large city. We will divide the long-term economically active adult inhabitants of the city into two groups. The first group consists of inhabitants who are economically active in the market as passive consumers and only occasionally share their experience regarding certain commodities with other consumers. Only by sharing their experience with commodities with other inhabitants do they affect the draw of retail trade to the large city, but they are less influential than the second group. The second group consists of adult inhabitants who in addition to passive participation are also active participants in the market exchange. This means that every member of this group of inhabitants intentionally maximizes his/her benefit from the exchange of goods on the market, both with respect to the quality and quantity of goods offered. This active market participation of the second group of inhabitants attracts the majority of retail activity to the large city (e.g. through the number of business loans, expressed in c.u. (currency units); targeted advertising in newspapers, which is quantitatively characterized by costs for advertising and distribution expressed in c.u. etc.).

Let there be two separate self-governing cities in Euclidean plane (cities \( A \) and \( B \)) with respective areas \( S_A \) and \( S_B \). The number of long-term economically active adult inhabitants in city \( A \) at time \( t \) is \( N_A \). The gravitational mass \( M_A \) of the given part of the population of city \( A \) at time \( t \) equals the sum gravitational masses \( m_{i,A} \) of the individual long-term economically active adult inhabitants, i.e.

\[
M_A = \sum_{i=1}^{N_A} m_{i,A}.
\]

Then \( M_A/N_A \) is the average gravitational mass \( \bar{m}_A \) of a single long-term economically active adult inhabitant in city \( A \) at time \( t \):

\[
\bar{m}_A = M_A / N_A = \left( \sum_{i=1}^{N_A} m_{i,A} \right) / N_A.
\]

The gravitational mass of all long-term economically active adult inhabitants in city \( A \) at time \( t \) is expressed by the average gravitational mass of a single long-term economically active adult inhabitant and the relationship
Methodology of Theoretical Physics in Economics: Vector Theory of Retail Gravitation Law

\[ M_A = \bar{m}_A N_A. \]  \hfill (5)

The average gravitational mass \( \bar{m}_g \) of a single long-term economically active adult inhabitant in city \( B \) at time \( t \) is expressed by the relationship

\[ \bar{m}_B = M_B / N_B = \left( \sum_{i=1}^{N_B} m_{i,B} \right) / N_B, \]  \hfill (6)

where

\[ M_B = \bar{m}_B N_B. \]  \hfill (7)

is the total gravitational mass of the number of long-term economically active adult inhabitants at the time \( t \) in city \( B \); \( N_B \) is the number of long-term economically active adult inhabitants in city \( B \) at time \( t \).

In the next step we will no longer consider the unequal density distribution of the long-term economically active adult inhabitants of cities \( A \) and \( B \). For simplicity of expression allowing the application of Newtonian physics, we will consider cities \( A \) and \( B \) to be mass points with masses \( M_A \) and \( M_B \). The magnitude of the gravitational force \( F_g \) between mass points \( A \) and \( B \) with gravitational masses \( M_A \) and \( M_B \) respectively is given by Newton’s Law of Gravitation \( F_g = \kappa M_A M_B / r_{AB}^2 \), where \( \kappa = 6.67408 \times 10^{-11} \) m \( ^3 \) kg \( ^{-1} \) s \( ^{-2} \) is the gravitational constant and \( r_{AB} \) is the distance between mass points \( A \) and \( B \) while Earth’ rotation is neglected.

The relationships \( M_A = \bar{m}_A N_A \) and \( M_B = \bar{m}_B N_B \) expressing the gravitational masses of the numbers of long-term economically active adult inhabitants of cities \( A \) and \( B \) are then substituted into Newton’s law of gravitation. Following this step, Newton’s law of gravitation has then the form

\[ F_g = \kappa \bar{m}_A \bar{m}_B N_A N_B / r_{AB}^2. \]  \hfill (8)

If we assume that \( \bar{m}_A = \bar{m}_B = \bar{m} \), then Newton’s law of gravitation acquires a simpler form

\[ F_g = \kappa \bar{m}^2 N_A N_B / r_{AB}^2. \]  \hfill (9)

From this last expression of Newton’s law of gravitation, we obtain the relationship

\[ \frac{F_g}{\kappa \bar{m}^2} = \frac{N_A N_B}{r_{AB}^2}, \]  \hfill (10)

which is the foundation of Reilly’s scalar law of retail gravitation.

3 Findings and results

In the 1930s the volume of retail trade attracted to intermediate town \( T \) was small and W. J. Reilly did not consider this in the theoretical part of his study. At the present time, the attraction of retail trade from the large city to intermediate town \( T \) is considered a common and economically significant phenomenon.

Let us assume that City \( A \) is represented on the geoid by a mass point, the gravitational mass \( M_A \) of which is equal to the sum gravitational masses of individual members of the city’s population, i.e. \( P_a \). For the reason that retail flows are realized in three dimensional space and time we define the potential of retail gravitation of the city \( A \) with the relation

\[ \phi(x, y, z) = \phi(r) = -\alpha \frac{P_a}{D_a} = -\alpha \frac{P_a}{\sqrt{x^2 + y^2 + z^2}}, \]  \hfill (11)
where \( D_a \) is the distance from City \( A \), \( D_b = \sqrt{x^2 + y^2 + z^2} \), \( \overrightarrow{D_a} = (x, y, z) \) is the position vector of the place of observation of retail trade volume with respect to mass point \( A \). The proportionality constant \( \alpha \) is expressed in units \([\alpha] = \text{c.u. m}^2 \text{ pers.}^{-2}\). Vector field \( \overrightarrow{K_a} \) of the intensity of retail trade gravitation is determined by the negative gradient of potential of retail gravitation \( \varphi \)

\[
\overrightarrow{K_a} = -\nabla \varphi .
\] (12)

If we express position vector \( \overrightarrow{D_a} \) through its components along the axes of the coordinate system, i.e. \( \overrightarrow{D_a} = xi + yj + zk \), then the intensity of retail trade gravitation is expressed in the form

\[
\overrightarrow{K_a} = -\alpha \frac{P_a}{D_a^3} \overrightarrow{D_a} = -\alpha \frac{P_a}{D_a^3} \left( -\alpha \frac{P_a}{D_a^3} \right) x \overrightarrow{i} + \left( -\alpha \frac{P_a}{D_a^3} \right) y \overrightarrow{j} + \left( -\alpha \frac{P_a}{D_a^3} \right) z \overrightarrow{k} .
\] (13)

This means that for the magnitude \( K_a \) of the vector of intensity of retail trade gravitation follows from the equation (13)

\[
K_a = |\overrightarrow{K_a}| = \alpha \frac{P_a}{D_a^2} .
\] (14)

Let \( A \) and \( B \) be two cities with large populations, which we mark \( P_a \) and \( P_b \). Let us assume there also exists intermediate town \( T \) with population \( P_t \), which is much smaller than the population of cities \( A \) and \( B \), i.e. \( P_t \ll P_a \) and \( P_t \ll P_b \). We mark the distances of cities \( A \) and \( B \) from town \( T \) as \( D_{at} \) and \( D_{bt} \). The law of retail gravitation for pairs of cities \( A, T \) and \( B, T \) is then in analytical form expressed by the following relations

\[
B_{at} = \alpha(T, A) \frac{P_a P_t}{D_{at}^2},
\] (15)

\[
B_{bt} = \alpha(T, B) \frac{P_b P_t}{D_{bt}^2},
\] (16)

where \( B_{at} \) is the business which City \( A \) draws from any given intermediate town \( T \) and \( B_{bt} \) is the business which City \( B \) draws from that intermediate town \( T \). In equation (15) \( \alpha(T, A) \) is the constant of proportionality for populations \( P_t \) and \( P_a \) of cities \( T \) and \( A \). In equation (16) \( \alpha(T, B) \) is the constant of proportionality for populations \( P_t \) and \( P_b \) of cities \( T \) and \( B \). In accordance with W. J. Reilly, we further assume that in equations (15) and (16) there is a universal constant of proportionality, i.e. \( \alpha(T, A) = \alpha(T, B) = \alpha \). It then follows

\[
\frac{B_{at}}{B_{bt}} = \frac{\alpha P_a P_t}{\alpha P_b P_t} \left( \frac{D_{at}}{D_{bt}} \right)^2 .
\] (17)

Quantities from equations (15) and (16) are expressed in basic units in the following manner: \([B_{at}] = [B_{bt}] = \text{c.u.} \) (currency unit), \([P_t] = [P_a] = [P_b] = \text{pers. (person)}\), \([D_{at}] = [D_{bt}] = \text{m} \) (meter). From equations (15) and (16) we get a dimensional equation for proportionality constant \( \alpha \) in Reilly’s law of retail gravitation

\[
c\text{u.} = [\alpha] \frac{\text{pers.}^3}{\text{m}^2}
\] (18)

from which we get

\[
[\alpha] = \text{c.u. m}^2 \text{ pers.}^{-2} .
\] (19)
Now let us define an equilibrium point for the retail gravitation through the vector intensity $\vec{K}$ of retail trade gravitation. Let $A$ and $B$ be two cities with large populations, which we mark $P_a$ and $P_b$. In a sufficiently small neighborhood of the equilibrium point between the two cities ($A$ and $B$), the vector intensity magnitude of retail gravitation $K_a$ for the city $A$ is equal to the vector intensity magnitude of retail gravitation $K_b$ for the city $B$, since the ratio $\frac{B}{B_b}$ at any equilibrium point is always equal to one or almost equal to one, i.e.

$$\frac{P_a}{D_a^2} = \frac{P_b}{D_b^2}$$ \hspace{1cm} (20)

and

$$D_a + D_b = D_{ab}, \hspace{1cm} (21)$$

where $D_{ab}$ is the total automobile highway distance between cities $A$ and $B$, $D_a$ is the total automobile highway distance of equilibrium point from the city $A$ and $D_b$ is the total highway automobile distance of equilibrium point from the city $B$. The equilibrium point lies on the shortest connecting line between the two cities attracting retail trade. The distance of the two cities is measured along the shortest highway connecting cities $A$ and $B$.

Under the assumption that $P_a = 160 \, 000$ inhabitants, $P_b = 250 \, 000$ inhabitants and distance between the two cities $D_{ab} = 150 \, km$ we get for the position of the equilibrium point on the shortest highway connecting cities $A$ and $B$ that $D_a = 66.7 \, km$ and $D_b = 83.3 \, km$.

4 Conclusion

In this work we only briefly outline the basic construction of three-dimensional vector theory of retail gravitation on a geoid. This means that we describe the course of economic processes in space and time with respect to the cosmic space near Earth (i.e. in Space Economics). Analysis of the relationship of vector theory of retail gravitation to other retail trade theories will be the subject of a separate article.

Acknowledgements

The author is grateful to Mrs. Pavla Jará and the National Library of Technology for their great effort and excellent work, which was indispensable in the completion of a large portion of this work. This paper is dedicated to Mrs. Věra Ruml Zeithamer, Mr. Josef Ruml Zeithamer, Mrs. Anna Ruml, Mr. František Ruml.

References
