On the Computation of B-Spline Basis Function Values $B_{Q,r}(t)$

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Abstract: In the article we describe the construction of B-spline basis functions $B_{Q,r}$ with the help of divided differences and determinants. The so-called Vandermonde determinant plays a fundamental role in expression for the particular determinants in our case. The computation of the value of $B_{Q,r}(t)$ at a given point $t \in \mathbb{R}$ can be realized through three formulas, the first of which has a general scope and the other two are its simplifications.

Key words: B-spline basis functions · de Boor formula · Vandermonde determinant

JEL Classification: C63 · C65

1 B-spline basis functions

By the symbol $(t)_+$ we denote the real-valued function

$$(t)_+ = \begin{cases} t, & \text{if } t > 0, \\ 0, & \text{if } t \leq 0. \end{cases}$$

A B-spline function $B_{Q,r} = B_{Q,r}(t)$ is defined for $Q \geq 1$ and $r$ integers, and $Q + 2$ knots $T_{r-q-1} < T_{r-q} < \cdots < T_r$ as a normalized $(Q+1)$-th divided difference of the function $g(T) = [(T - t)_+]^Q$ of real variable $T$. Thus, $g(T)$ is, for a given $T$, function of the real variable $t$, which we will denote as $(T - t)_+^Q$. Hence,

$$B_{Q,r} = (T_r - T_{r-q-1})[T_{r-q-1}, T_{r-q}, \ldots, T_r]g.$$  \hfill (1)

(Divided differences can be found e.g. in (Schrutka, 1945), for spline-related topics we refer to the other literature listed at the end of the work.)

For example, for $Q = 1$ we have, according to (1),

$$B_{1,r} = (T_r - T_{r-2}) \cdot [T_{r-1}, T_{r-2}, T_r]g = (T_r - T_{r-2}) \frac{[T_r, T_{r-1}]g - [T_{r-1}, T_{r-2}]g}{T_r - T_{r-2}} =$$

$$= [T_r, T_{r-1}]g - [T_{r-1}, T_{r-2}]g = \frac{g(T_r) - g(T_{r-1})}{T_r - T_{r-1}} - \frac{g(T_{r-1}) - g(T_{r-2})}{T_{r-1} - T_{r-2}} =$$

$$= \frac{(T_r - t)_+^1 - (T_{r-1} - t)_+^1}{T_r - T_{r-1}} + \frac{(T_{r-2} - t)_+^1 - (T_{r-1} - t)_+^1}{T_{r-1} - T_{r-2}},$$

that is,

$$B_{1,r}(t) = \begin{cases} \frac{T_r - t}{T_r - T_{r-1}}, & \text{for } T_{r-1} \leq t \leq T_r, \\ 0, & \text{elsewhere}. \end{cases}$$  \hfill (2)

$$B_{1,r}(t) = \begin{cases} \frac{t - T_{r-2}}{T_{r-1} - T_{r-2}}, & \text{for } T_{r-2} \leq t \leq T_{r-1}, \\ 0, & \text{elsewhere}. \end{cases}$$  \hfill (3)

everywhere else $B_{1,r}(t)$ takes the value zero.

2 General expression of divided differences through determinants

In (1), the following expression incorporating determinants holds for $[T_{r-q-1}, T_{r-q}, \ldots, T_r]g$ (see Schrutka, 1945), for example):

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where both determinants are of order \( Q + 2 \), and the determinant in the denominator of this fraction is a Vandermonde determinant

\[
V_{Q+2}(T_{r-q-1}, T_{r-q}, \ldots, T_r) = (-1)^\sigma \prod_{0 \leq q < Q} \prod_{1 \leq s < Q+1-q} (T_{r-q} - T_{r-q-s}),
\]

where \( \sigma = \lfloor Q/2 \rfloor + 1 \) (\( \lfloor x \rfloor \) denotes the whole part of the real number \( x \)).

For example, for \( Q = 1 \) it follows from (1), (4) that

\[
B_{1,r} = \left( \frac{T_r - T_{r-2}}{V_3(T_{r-2}, T_{r-1}, T_r)} \right) \frac{V_2(T_{r-1}, T_r)}{V_3(T_{r-2}, T_{r-1}, T_r)} (T_r - t) + (T_{r-1} - t),
\]

whereas by expanding the determinant in the numerator of this fraction we get that

\[
B_{1,r} = \left( \frac{T_r - T_{r-2}}{V_3(T_{r-2}, T_{r-1}, T_r)} \right) \frac{V_2(T_{r-1}, T_r)}{V_3(T_{r-2}, T_{r-1}, T_r)} (T_r - t) + (T_{r-1} - t),
\]

where, according to (5),

\[
V_3(T_{r-2}, T_{r-1}, T_r) = (-1)^{\lfloor 0.5 \rfloor + 1} (T_{r} - T_{r-1})(T_{r} - T_{r-2})(T_{r-1} - T_{r-2}),
\]

and according to the same formula, if we substitute \( Q \) by \( Q - 1 \) and \( r \) by \( r - 1 \), we arrive to (again for \( Q = 1 \)) the expression \( V_2(T_{r-1}, T_r) = (-1)^{\lfloor 0.5 \rfloor + 1} (T_{r-1} - T_{r-2}) \). And hence

\[
B_{1,r} = \left( \frac{T_r - T_{r-2}}{V_3(T_{r-2}, T_{r-1}, T_r)} \right) \frac{V_2(T_{r-1}, T_r)}{V_3(T_{r-2}, T_{r-1}, T_r)} (T_r - t) + (T_{r-1} - t),
\]

for \( T_{r-1} \leq t \leq T_r \), which is the same result as (2). Analogously, we may prove the validity of (3) for \( T_{r-2} \leq t \leq T_{r-1} \).

Example 1. For \( Q = 2, r = 5 \) and knots \( T_2 = 2, T_3 = 4, T_4 = 6, T_5 = 10 \) there is, according to (7),

\[
B_{2,5}(t) = \sum_{q=0}^{Q+1} \left( \prod_{0 \leq q < Q+1-q} (T_{r-q} - t) \right) \prod_{s=r-q+1} \left( \prod_{0 \leq q < Q+1-q} (T_{r-q} - T_{r-s}) \right).
\]

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recall that \((T - t)^2 = [(T - t)_+]^2\). For example, for \(t = 5 \in \langle 4, 6 \rangle = \langle T_3, T_4 \rangle\) it follows that
\[
B_{2,5}(5) = \frac{25}{192} - \frac{1}{32} = \frac{19}{24} = 0.7917.
\]

3 Specifying the obtained expression for the computation of B-spline values

It is well known that B-spline basis functions \(B_{Q,r}\) possess many interesting properties, see (Meloun & Militký, 1994) for example. We state four of them.

a) They are positive only in the intervals \(T_{r-2} < t < T_r\) and are zero everywhere else.
b) They are normalized, that is, for \(r = 1, 2, \ldots, k + 1\),
\[
\sum_{r=1}^{k+Q+1} B_{Q,r}(t) = 1
\]

in \(\langle T_0, T_{k+1} \rangle\); for a complete definition of B-splines in the sum (13) we need to set on every side of that interval another \(Q\) so-called complementary knots
\[
T_{-Q} < T_{-Q+1} < \ldots < T_{-1} < T_0, \quad T_{k+1} < T_{k+2} < \ldots < T_{k+Q+1},
\]
in the simplest case they merge with \(T_0\) and \(T_{k+1}\) on the left and right side. We call \(T_1 < T_2 < \ldots < T_k\), where \(T_0 < T_1\) and \(T_k < T_{k+1}\), the main knots.
c) In every interval \(\langle T_{p-1}, T_p \rangle, p = 1, 2, \ldots, k + 1\), exactly \(B_{Q,p}, B_{Q,p+1}, \ldots, B_{Q,p+Q}\) are non-zero, altogether \(Q + 1\) in number.
d) \(B_{Q,r}\) is in \(\langle T_{r-Q-1}, T_r \rangle\) a polynomial spline of order \(Q\) with knots \(T_{r-Q-1} < T_{r-Q} < \ldots < T_r\), that is, in every closed interval defined by two neighboring points \(B_{Q,r}\) is a polynomial of order \(Q\) that belongs to the class \(C^{Q-1}(T_{r-Q-1}, T_r)\).

As an example, for \(Q = 1, k = 2\) and main knots \(T_1 = 1, T_2 = 3\) with complementary knots \(T_{-1} = -1, T_0 = 0\) and \(T_3 = 6, T_4 = 9\), we give below the B-spline basis functions for \(r = 1, 2, 3, 4 = k + Q + 1\) (we used (2) and (3) for their derivation):

\[
B_{1,1}(t) = \begin{cases} 
+t+1 & \text{for } -1 \leq t \leq 0, \\
1-t & \text{for } 0 \leq t \leq 1, \\
0 & \text{otherwise},
\end{cases}
\]

\[
B_{1,2}(t) = \begin{cases} 
t & \text{for } 0 \leq t \leq 1, \\
-\frac{1}{2}(t-3) & \text{for } 1 \leq t \leq 3, \\
0 & \text{otherwise},
\end{cases}
\]

\[
B_{1,3}(t) = \begin{cases} 
\frac{1}{2}(t-1) & \text{for } 1 \leq t \leq 3, \\
-\frac{1}{3}(t-6) & \text{for } 3 \leq t \leq 6, \\
0 & \text{otherwise},
\end{cases}
\]

\[
B_{1,4}(t) = \begin{cases} 
\frac{1}{3}(t-3) & \text{for } 3 \leq t \leq 6, \\
-\frac{1}{3}(t-9) & \text{for } 6 \leq t \leq 9, \\
0 & \text{otherwise}.
\end{cases}
\]

We easily verify that the aforementioned properties a), b), c) and d) hold.

Example 2. For \(Q = 2, k = 2\) and main knots \(T_1 = 1, T_2 = 3\) with complementary knots \(T_{-2} = -2, T_{-1} = 1, T_0 = 0\), and \(T_3 = 6, T_4 = 9, T_5 = 12\) (the number of knots is equal to \(k + 2(Q + 1) = 8\)), we may, through the de Boor formula, see (de Boor, 1972),
\[
B_{Q+1,r} = \frac{T_r - T_{r-Q-2}}{T_{r-Q-1} - T_{r-Q-2}} B_{Q,r-1} + \frac{T_r - t}{T_r - T_{r-Q-1}} B_{Q,r},
\]

get the explicit expression, e.g., for \(B_{2,4}\) together with the help of the results (9) (note that in (10) we put \(Q = 1\)):
In this way we get the following expression:

\[
B_{2,4}(t) = \begin{cases} 
\frac{t - 11t - 1}{2} + \frac{9 - t}{6} & \text{for } 1 \leq t \leq 3, \\
\frac{t - 16 - t + 9 - t}{3} + \frac{9 - t - 3}{6} & \text{for } 3 \leq t \leq 6, \\
\frac{t - 9 - t + 9 - t}{3} + \frac{9 - t}{6} & \text{for } 6 \leq t \leq 9, \\
0 & \text{otherwise.}
\end{cases}
\]

For example, for \( t = 5 \in (3, 6) = (T_2, T_3) \) there is \( B_{2,4}(5) = 0.7111 \).

According to the aforementioned property c), in the interval \((T_{p-1}, T_p)\), \( p = 1, 2, ..., k + 1 \), are the only non-zero functions \( B_{q,p}, B_{q,p+1}, ..., B_{q,p+Q} \). Using this fact, for \( \rho = 0, 1, ..., Q \), it follows from (7) through an easy computation that

\[
B_{q,p+\rho}(t) = (T_{p+\rho} - T_{p+\rho-Q+1}) \sum_{q=0}^{\rho} \left( \frac{\rho}{q} \right) \prod_{s=q}^{Q} \left( T_{p+\rho-q} - T_{p+\rho-s} \right),
\]

for \( t \in (T_{p-1}, T_p) \).

For example, for \( Q = 2, k = 2 \) and the increasing sequence of knots from Example 2, it follows from (11) that, for \( p = 3, \rho = 1 \),

\[
B_{2,4}(t) = (T_4 - T_1) \sum_{q=0}^{3} \left( \frac{9-1}{3} \right) + \prod_{s=q}^{4} \left( T_{4-q} - T_{4-s} \right) = 8 \left( \frac{9-t}{3} \right) + \left( \frac{6-t}{6} \right) = 8 \left( \frac{9-t}{144} \right) - \frac{6-t}{45} = \frac{8 \cdot (9-t)}{720} - \frac{16 \cdot (6-t)}{720},
\]

for \( t \in (T_{p-1}, T_p) = (T_2, T_3) = (3, 6) \). For example, for \( t = 5 \in (3, 6) \), there is

\[
B_{2,4}(5) = \frac{80 - 16}{720} = \frac{32}{45} = 0.7111,
\]

which is in agreement with the result of Example 2.

**Example 3.** For \( Q = 3, k = 1 \), main knots \( T_1 = 1, T_0 = 0 \) with complementary knots \( T_{-2} = -4, T_{-1} = -2, T_1 = 1, T_0 = 0, \) and \( T_2 = 3, T_3 = 6, T_4 = 9, T_5 = 12 \) (the number of knots is equal to \( k + 2(Q + 1) = 9 \)), there is for \( p = 2, \rho = 3, \) and \( t \in (T_{p-1}, T_p) = (T_1, T_2) = (1, 3) \), according to (11),

\[
B_{3,5}(t) = (T_5 - T_1) \sum_{q=0}^{3} \left( \frac{9-1}{3} \right) + \prod_{s=q}^{5} \left( T_{5-q} - T_{5-s} \right) = 11 \left( \frac{12-t}{12} \right) + \left( \frac{6-t}{6} \right) + \left( \frac{3-t}{3} \right) = \frac{11 \cdot 1782}{71280} - \frac{432}{71280} + \frac{324}{71280} = \frac{40 \cdot (12-t)}{71280},
\]

For example, for \( t = 2 \in (1, 3) \) there is

\[
B_{3,5}(2) = 11 \frac{40 \cdot 1000 - 165 \cdot 343 + 264 \cdot 64 - 220 \cdot 1}{71280} = \frac{891}{71280} = 0.0125.
\]

In the relation (11) one might set \( p + \rho = r \), that is, \( \rho = r - p \), while \( 0 \leq \rho = r - p \leq Q \), which means that

\[
p \leq r \leq Q + p.
\]

In this way we get the following expression:
\[ B_{Q,r}(t) = (T_r - T_{r-q-1}) \sum_{q=0}^{r-p} \frac{(T_{r-q} - t)^{q}}{\prod_{0 \leq s \leq q+1 \neq r} (T_{r-q} - T_{r-s})} \]  

(13)

for \( t \in (T_{r-1}, T_r) \).

Example 4. Let \( Q = 4, k = 6 \), let the main knots be \( T_i = 3(i + 1) \) for \( i = 1, 2, ..., 6 \), and the complementary knots then \( T_{-5+j} = 3(j - 4) \) for \( j = 1, 2, ..., 5 \) and \( T_{6+l} = 3(7 + l) \) for \( l = 1, 2, ..., 5 \). The increasing sequence of knots

\[ T_{-4} = -9 < -6 = T_{-3} < T_{-2} = -3 < ... < T_0 = 3 < 6 = T_1 < T_2 = 9 < ... < T_{10} = 33 < 36 = T_{11} \]

consists of \( k + 2(Q + 1) = 16 \) elements. For example, for \( r = 10, r = 7 \), there will be, according to (13), in the interval \( (T_{p-1}, T_p) = (T_6, T_7) = (21, 24) \) (note that condition (12) is satisfied, as \( 7 \leq 10 \leq 11 \))

\[ B_{8,10}(t) = (T_{10} - T_5) \sum_{q=0}^{3} \frac{(T_{10-q} - t)^{4}}{\prod_{0 \leq s \leq 4 \neq 10} (T_{10-q} - T_{10-s})} = \]

\[ = 15 \left\{ \frac{(33 - t)^4}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15} + \frac{(30 - t)^4}{(-3) \cdot 3 \cdot 6 \cdot 9 \cdot 12} + \frac{(27 - t)^4}{(-6) \cdot (-3) \cdot 3 \cdot 6 \cdot 9} + \frac{(24 - t)^4}{(-9) \cdot (-6) \cdot (-3) \cdot 3 \cdot 6} \right\} = \]

\[ = \frac{1}{1944} (33 - t)^4 - 5 \cdot (30 - t)^4 + 10 \cdot (27 - t)^4 - 10 \cdot (24 - t)^4 \].

For example, for \( t = 23 \in (24, 27) \) there will be

\[ B_{8,10}(23) = \frac{1}{1944} (10^4 - 5 \cdot 7^4 + 10 \cdot 4^4 - 10 \cdot 1^4) = \frac{545}{1944} = 0.2803. \]

4 Conclusion

The first one of the three formulas (7), (11), (13) for the computation of B-spline basis function values \( B_{Q,r}(t) \) at given point \( t \in \mathbb{R} \) has general validity, while the other two are its simplifications when taking into account the property c) of basis functions, see p. 3 of this work. The author of this article developed a computer program for the evaluation of the values \( B_{Q,r}(t) \), as the computation of these by hand would have obviously been too laborious.

References